L1 - Determinacy

20 October 2026 23:50

If $\langle e,s\rangle \to \langle e',s'\rangle$ and $\langle e,s\rangle \to \langle e'',s''\rangle$
then $e' = e''$ and $s' = s''$.

$\Phi(e) = \forall e', e'', s', s'', s.\ (\langle e,s\rangle \to \langle e',s'\rangle \wedge \langle e,s\rangle \to \langle e'',s''\rangle) \to (e' = e'' \wedge s' = s'')$

Show $\forall e.\ \Phi(e)$ by induction over term structure.

Case(s) $e = n, b,$ skip

No reduction $\langle e, \ldots\rangle \to \langle \ldots\rangle$ so false $\to$ (...) holds

Case $e = !l$

Assume $\langle !l, s\rangle \to \langle e', s'\rangle$ and $\langle !l, s\rangle \to \langle e'', s''\rangle$

Only applicable rule is (deref) so we have

$l \in dom(s),\ e' = s(l),\ s' = s,\ e'' = s(l),\ s'' = s$

and so $e' = e'' \wedge s' = s''$

Case $l := e_0$, suppose $\Phi(e_0)$

Assume $\langle l := e_0, s\rangle \xrightarrow{①} \langle e', s'\rangle$ and $\langle l := e_0, s\rangle \xrightarrow{②} \langle e'', s''\rangle$

Case ① is (assign1)

Then $e_0 = n, l \in dom(s),\ e' = skip,\ s' = s \cup \{l \mapsto n\}$

Case ② is (assign1)

Then $e'' = skip,\ s'' = s \cup \{l \mapsto n\}$ — holds

Case ② is (assign2)

From ① $e = n$ but n is a value and cannot be reduced — contradiction

Case ① is (assign2)

Then $\langle e_0, s\rangle \to \langle e_0', s'\rangle$

Case ② is (assign2)

Then $\langle e_0, s\rangle \to \langle e_0'', s''\rangle$

By induction $e_0' = e_0''$ and $s' = s''$

So $e' = l := e_0'$ = $e'' = l := e_0''$ and $s' = s''$

Case ① is (assign1)

Then $e_0 = n$ for some n but n cannot be reduced — contradiction

Case e_1 op e_2, suppose $\Phi(e_1) \wedge \Phi(e_2)$

Assume $\langle e_1 \text{ op } e_2, s\rangle \xrightarrow{①} \langle e', s'\rangle$ and $\langle e_1 \text{ op } e_2, s\rangle \xrightarrow{②} \langle e'', s''\rangle$

By examining the rules applicable to $\langle e_1 \text{ op } e_2, s\rangle$
the only possible combinations are:

Case ① (op p) then $e_1 = n_1,\ e_2 = n_2,\ s' = s,\ e' = n$ (where $n = n_1 + n_2$)

② (op p) also $s'' = s,\ e'' = n$ — holds

Case ① (op gt) then $e_1 = n_1, e_2 = n_2,\ s' = s,\ e = true$ $(n_1 \geq n_2)$

② (op gt) also $s'' = s,\ e'' = true$ — holds

Case ① (op lt), ② (op lt) — similar

Case ① (op1) then $\langle e_1, s\rangle \to \langle e_1', s'\rangle,\ e' = e_1' \text{ op } e_2$

② (op1) then $\langle e_1, s\rangle \to \langle e_1'', s''\rangle,\ e'' = e_1'' \text{ op } e_2$

By $\Phi(e_1)$ $e_1' = e_1'' \wedge s_1' = s_1''$ — holds

Case ① (op2), ② (op2) — similar

RECALL

Operations	op := + | ≥
Expressions	e := n | b | e op e | if e then e else e | while e do e | l := e | !e | skip | e ; e

HELPER

Lemma: if e = skip, n or b then $\forall s\ \nexists e', s'.\ \langle e,s\rangle \to \langle e', s'\rangle$

Proof by inspection of the rules for $\to$

Cases $e_1; e_2$, while e_1 do e_2, if e_1 then e_2 else e_3
similar (see Cambridge notes for more detail)