L1 - Preservation

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If $\Gamma \vdash e : \tau$ and $dom(\Gamma) \subseteq dom(s)$ and $\langle e, s\rangle \to \langle e', s'\rangle$ then $\Gamma \vdash e' : \tau$ and $dom(\Gamma) \subseteq dom(s')$

For the second part, show if $\langle e, s\rangle \to \langle e', s'\rangle$ then $dom(s') = dom(s)$ — by induction over $\to$

$\Phi(e, s, e', s') = \forall \Gamma, \tau . (\Gamma \vdash e : \tau \wedge dom(\Gamma) \subseteq dom(s)) \Rightarrow \Gamma \vdash e' : \tau$

Show $\Phi(e, s, e', s')$ by induction over derivation $\langle e, s\rangle \to \langle e', s'\rangle$

Case (op p) $e = n_1 + n_2$, $\langle n_1 + n_2, s\rangle \to \langle n, s\rangle$

Assume Γ, τ such that $\Gamma \vdash e : \tau \wedge dom(\Gamma) \subseteq dom(s)$

Then last rule of $\Gamma \vdash e : \tau$ must have been (op +)

and so $\tau = int$

Since $\Gamma \vdash n : int$ by (int) also — holds

Case (op 1) $e = e_1 \text{ op } e_2$

and:
$$\frac{\langle e_1, s\rangle \to \langle e_1', s'\rangle}{\langle e_1 \text{ op } e_2, s\rangle \to \langle e_1' \text{ op } e_2, s'\rangle} \text{ (op 1)}$$

assume $\Phi(e_1, s, e_1', s')$

show $\Phi(e_1 \text{ op } e_2, s, e_1' \text{ op } e_2, s')$

suppose $\Gamma \vdash e_1 \text{ op } e_2 : \tau$, $dom(\Gamma) \subseteq dom(s)$

Case op = +

Last rule (op +), $\tau = int$, $\Gamma \vdash e_1 : int$, $\Gamma \vdash e_2 : int$

From $\Phi(e_1, s, e_1', s')$ have $\Gamma \vdash e_1' : int$

Therefore $\Gamma \vdash e_1' + e_2 : int$ (op +)

Case op = $\geq$ similar

Remaining cases

Similar, see Cambridge notes for more detail